Graphs

- Definition
  - A graph $G = (V, E)$ consists of a finite set of vertices, $V$, and a finite set of edges, $E$.

- Properties
  - $V$ and $E$ are sets: each vertex $v \in V$ and each edge $e \in E$ are unique.
  - Edge: $(v, w)$, where $v, w \in V$
  - Vertices and edges may store elements.

Example:
- A vertex represents an airport and stores the three-letter airport code.
- An edge represents a flight route between two airports and stores the mileage of the route.

Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram
Basic Graph Definitions

- **Directed graph (= digraph)**
  - Graph with directed edges
  - Directed edges
    - Ordered pair of vertices, $(v, w) \neq (w, v)$
    - The first vertex is the origin and the second vertex is the destination.

- **Undirected graph**
  - Graph with undirected edges
  - Undirected edges: $(v, w) = (w, v)$

- **Sparse graph**
  - Graph with few edges
    - $|E| = O(|V|)$, where $|\cdot|$ is the number of elements in a set

- **Dense graph**
  - Graph with many edges: $|E| = O(|V|^2)$

**Undirected Graph**

- $V = \{1, 2, 3, 4, 5\}$
- $E = \{(1,2), (2,3), (3,4), (4,5), (5,1), (2,4)\}$

**Directed Graph**

- $V = \{1, 2, 3, 4, 5\}$
- $E = \{(1,2), (3,2), (4,3), (4,5), (5,4), (5,1), (2,4)\}$
Terminology

- **End vertices (or endpoints) of an edge**
  - \( u \) and \( v \) are the endpoints of \( e_a \).

- **Edges incident on a vertex**
  - \( e_a, e_b, \) and \( e_d \) are incident on \( v \).

- **Adjacent vertices**
  - \((u, v) \in E\)
  - Vertices \( u \) and \( v \) are adjacent.

- **Degree of a vertex**
  - Number of incident edges
  - \( \deg(x) = 5 \)

- **Self-loop**
  - Edge between the same vertex
  - \( e_j \) is a self-loop.

Terminology (Cont.)

- **Path**
  - Sequence of vertices \( v_1, v_2, ..., v_n \) such that \((v_i, v_{i+1}) \in E\)
  - Length of a path
    - Number of edges on the path
    - The length of the path from a vertex to itself is 0.

- **Simple path**
  - Path such that all its vertices and edges are distinct
  - Example: \( p_1 = (v, x, z) \)
  - Non-simple path: \( p_2 = (u, w, y, y, w, v) \)

Terminology in Directed Graph

- **Cycle**
  - Sequence of vertices \( v_1, v_2, ..., v_n \) such that \((v_i, v_{i+1}) \in E\) and \( v_1 = v_n \)
  - No backtracking

- **Simple cycle**
  - Cycle such that all its vertices and edges are distinct.
  - Example: \( c_1 = (v, x, y, w, u, v) \)
  - Non-simple cycle example: visiting the same node multiple times
  - \( c_2 = (u, w, x, y, w, v, u) \)

- **Adjacency**
  - For directed graphs, vertex \( w \) is adjacent to \( v \) if and only if \((v, w) \in E\).

- **Degree**
  - Indegree
    - Number of incoming edges
    - \( \deg_{in}(X) = 2 \)
  - Outdegree
    - Number of outgoing edges
    - \( \deg_{out}(W) = 2 \)

- **Path and cycle in directed graph**
  - Should consider the direction of edges
  - Directed Acyclic Graph (DAG): directed graph with no cycles
Properties of Graphs

- For undirected graph
  - $G = (V, E)$, where $|V| = n$ and $|E| = m$
  - $\sum_v \deg(v) = 2m$
    - Each edge is counted twice.
  - $m \leq n(n-1)/2$ with no self-loops and no multiple edges
    - Each vertex has degree at most $(n-1)$

- What is the bound for a directed graph?

Connectivity

- Connected graph
  - In undirected graph
    - There is a path from every vertex to every other vertex.
  - Strongly connected graph
    - There is a path from every vertex to every other vertex.
  - Weakly connected graph
    - In directed graphs
      - There is a path from every vertex to every other vertex, disregarding the direction of the edges.

Subgraph

- A graph $S$ is a subgraph of $G$ if and only if $V_S \subseteq V_G$ and $E_S \subseteq E_G$.

Complete graph

- There is an edge between every pair of vertices.
- Connected component: maximal connected subgraph

Non-connected graph with two connected components